

Stats 369 Project

Yihan Lai Puheng Li Ruochuan Xu

December 2025

Abstract

This report reviews the Langevin dynamic approach to the lattice Yang-Mills problem, centered around the paper [SZZ23]. We explore various aspects of the construction of stochastic quantization for this theory with compact structure groups $SO(N)$ and $SU(N)$, and discuss applications of the exponential ergodicity of the Langevin dynamics, including the large- N factorization of Wilson loops and the existence of a mass gap.

1 Introduction

The rigorous mathematical construction of quantum Yang-Mills theory in four dimensions remains a central problem in mathematical physics. Initiated by the work of Kenneth Wilson [Wil74], lattice gauge theory presents a non-perturbative regularization of the heuristic continuum theory of gauge fields by discretizing spacetime onto a lattice. The paper “A stochastic analysis approach to lattice Yang-Mills at strong coupling” by Hao Shen, Rongchan Zhu, and Xiangchan Zhu [SZZ23] introduces a dynamical perspective to the study of such lattice systems. While much of the literature on this theory focuses on combinatorial analysis of the measure using techniques such as cluster expansion, the present paper considers Langevin dynamics that converge to the lattice Yang-Mills measure. For simplicity, the analysis focuses on the compact gauge groups $SO(N)$ and $SU(N)$. By studying the convergence of Langevin dynamics using functional inequalities and techniques at the intersection of Riemannian geometry and stochastic analysis, the authors are able to establish in the strong coupling regime the uniqueness of infinite-volume invariant measure. With this key result and the detailed behaviors of the Langevin dynamics unraveled in its proof, they further derive novel as well as previously known results in lattice Yang-Mills theory, such as mass gap and large- N factorization of Wilson loop expectations under strong coupling.

2 Main ingredients

2.1 The measure

Given a structure group G usually taken to be a matrix Lie group in $M_N(\mathbb{C})$ and a lattice $\Lambda \subset \mathbb{Z}^d$, the lattice Yang-Mills theory is the study of certain probability measures on the configuration

space $\mathcal{Q}_\Lambda := G^{E_\Lambda^+}$, where E_Λ^+ denotes the set of positively oriented edges in Λ . For an inverse coupling parameter $\beta \in \mathbb{R}$, the lattice Yang-Mills measure $\mu_{\Lambda,N,\beta}$ has density with respect to the product Haar measure proportional to $\exp(S_{\Lambda,N,\beta}(Q))$ at a configuration $Q = (Q_e)_{e \in E_\Lambda^+}$ of G -matrices. Here, the lattice Yang-Mills action $S_{\Lambda,N,\beta}(Q)$ is given by

$$S_{\Lambda,N,\beta}(Q) := N\beta \operatorname{Re} \sum_{p \in \mathcal{P}_\Lambda^+} \operatorname{Tr}(Q_p), \quad (1)$$

where $\mathcal{P}_\Lambda^+$ denotes the set of positively oriented plaquettes (i.e. four-sided squares) in Λ and $Q_p := Q_{e_1}Q_{e_2}Q_{e_3}Q_{e_4}$ for a plaquette $p = e_1e_2e_3e_4$. Denoting the Haar measure on G by σ_N , we can write out the measure more explicitly with a normalizing constant $Z_{\Lambda,N,\beta}$ as

$$d\mu_{\Lambda,N,\beta}(Q) := Z_{\Lambda,N,\beta}^{-1} \exp(S_{\Lambda,N,\beta}(Q)) \prod_{e \in E_\Lambda^+} d\sigma_N(Q_e). \quad (2)$$

The measure $\mu_{\Lambda,N,\beta}$ is well-defined for a finite $\Lambda \subset \mathbb{Z}^d$. In particular, we consider d -dimensional hypercubes Λ_L in $\mathbb{Z}^d$ of sidelength L , and denote the corresponding configuration space by $\mathcal{Q}_L := G^{\Lambda_L}$. To make sense of the lattice Yang-Mills measure over the infinite-volume configuration space $\mathcal{Q} := G^{\mathbb{Z}^d}$, one may obtain it as the unique weak limit of $\mu_{\Lambda_L,N,\beta}$ as $L \rightarrow \infty$, as considered in works such as [Cha17]. By contrast, [SZZ23] distinguishes the infinite-volume measure $\mu_{N,\beta}^{\text{YM}}$ as the unique invariant measure of the infinite-volume Langevin dynamic. This characterization of the infinite-volume lattice Yang-Mills measure is one of the main contributions of the paper.

2.2 The geometry

An essential viewpoint of [SZZ23] is the treatment of the state space $\mathcal{Q}_L$ as a Riemannian manifold; although $\mathcal{Q}$ is not a Riemannian manifold per se, key objects such as the differential operators and distance of finite product Riemannian manifolds readily generalizes when the product is infinite. This geometrical perspective sets the framework for the construction and analysis of the Langevin dynamics.

2.2.1 Riemannian metric

A matrix Lie group $G \subset M_N(\mathbb{C})$ with Lie algebra $\mathfrak{g}$ is a $d(\mathfrak{g}) := \dim \mathfrak{g}$ -dimensional Riemannian manifold with a bi-invariant metric given by the Hilbert-Schmidt inner product

$$\langle X, Y \rangle = \operatorname{Re} \operatorname{Tr}(XY^*) \quad \text{for } X, Y \in \mathfrak{g}.$$

That is, every $X \in \mathfrak{g}$ gives a right-invariant vector field that takes value XQ in $T_Q G$, so that the bi-invariance of the inner product follows from $\operatorname{Tr}((XQ)(YQ)^*) = \operatorname{Tr}(XY^*)$.

With the Riemannian structure on G , we may consider $\mathcal{Q}_L = G^{\Lambda_L}$ as a product Riemannian manifold, with the metric given by the sum of the metric of each component: for $X, Y \in \mathfrak{g}_L := \mathfrak{g}^{E_{\Lambda_L}^+}$,

$$\langle XQ, YQ \rangle_{T_Q \mathcal{Q}_L} = \sum_{e \in E_{\Lambda_L}^+} \langle X_e Q_e, Y_e Q_e \rangle_{T_{Q_e} G} = \sum_{e \in E_{\Lambda_L}^+} \operatorname{Tr}(X_e Y_e^*).$$

2.2.2 Differential operators

At each $e \in E_{\Lambda_L}^+$, we can consider an orthonormal basis of $T_{Q_e} G$ denoted by $\{v_e^i : i = 1, \dots, d(\mathfrak{g})\}$. Then the usual basis-independent gradient and Laplace operators on the Riemannian manifold G in the e -component of $\mathcal{Q}_L$ can be written as

$$\nabla_e f := \sum_{i=1}^{d(\mathfrak{g})} (v_e^i f) v_e^i, \quad \Delta_e f := \operatorname{div} \nabla_e f = \sum_{i=1}^{d(\mathfrak{g})} \langle \nabla_{v_e^i} \nabla_e f, v_e^i \rangle$$

We then obtain the gradient and Laplace operators on the product Riemannian manifold $\mathcal{Q}_L$ as

$$\nabla_L := \sum_{e \in E_{\Lambda_L}^+} \nabla_e \quad \Delta_L := \sum_{e \in E_{\Lambda_L}^+} \Delta_e. \quad (3)$$

Similarly, denote the infinite volume operators obtained by summing over $e \in E^+$ as ∇ and Δ .

2.2.3 Riemannian distance

Let ρ be the standard Riemannian distance on G , then the induced distance on the product Riemannian manifold $\mathcal{Q}_L$ is given by

$$\rho_L(Q, Q')^2 := \sum_{e \in E_{\Lambda_L}^+} \rho(Q_e, Q'_e)^2 \quad Q, Q' \in \mathcal{Q}_L$$

For probability measures μ, ν on $\mathcal{Q}_L$, this induces the p -Wasserstein distance

$$W_p^{\rho_L}(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \pi(\rho_L^p)^{1/p},$$

where $\mathcal{C}(\mu, \nu)$ denotes the set of couplings between μ and ν . In infinite volume, we consider

$$\rho_{\infty, a}^2(Q, Q') := \sum_{e \in E^+} \frac{1}{a^{|e|}} \rho^2(Q_e, Q'_e),$$

where $|e|$ is the distance of e to the origin, and $a > 1$ is some real parameter. It turns out that the normalizing factor $a^{-|e|}$ not only guarantees finiteness of $\rho_{\infty, a}$, but the parameter a also appears in the speed of exponential contraction of the associated 2-Wasserstein distance $W_2^{\rho_{\infty, a}}(\cdot, \cdot)$, so we need to choose a to be sufficiently close to one if we want this speed to be positive. Note that the distance $\rho_{\infty, a}$ in infinite volume is defined somewhat arbitrarily, but the resulting topology is always equivalent to the product topology on $\mathcal{Q}$.

2.3 The dynamics

Recall that the lattice Yang-Mills measure has density proportional to $e^{S(Q)}$. When the state space is a finite-dimensional manifold, a Langevin dynamic can be canonically constructed to leave a measure of this form invariant. Thanks to the locality of the interaction in the lattice Yang-Mills action $S(Q)$, it turns out that the Langevin dynamic in infinite volume can be simply defined to agree locally with the finite volume Langevin dynamic, although the analysis of the resulting SDE system requires much technical care.

2.3.1 finite volume Langevin dynamics

The Langevin dynamic for the lattice Yang-Mills measure (2) is the SDE on $\mathcal{Q}_L$ given by

$$dQ = \nabla_L S(Q) dt + \sqrt{2} d\mathfrak{B}, \quad (4)$$

where $\mathfrak{B} = (\mathfrak{B}_e)_{e \in E_{\Lambda_L}^+}$ are independent Brownian motions on G . Brownian motions on Lie groups are made sense of through Brownian motions on corresponding Lie algebras, and the SDE can be explicitly written in each component e as

$$dQ_e = \nabla_L S(Q)_e dt + c_{\mathfrak{g}} Q_e dt + \sqrt{2} dB_e Q_e, \quad (5)$$

where $c_{\mathfrak{g}}$ is the Casimir constant explicitly given for the Lie groups under consideration by

$$c_{\mathfrak{so}(N)} = -\frac{1}{2}(N-1) \quad c_{\mathfrak{su}(N)} = -\frac{N^2-1}{N}.$$

Moreover, using $p \succ e$ to denote oriented plaquettes p that start from $e \in E_{\Lambda_L}^+$, $\nabla_L S(Q)_e$ can be explicitly computed to be

$$\nabla_L S(Q)_e = \begin{cases} -\frac{1}{2}N\beta \sum_{\substack{p \in \mathcal{P}_{\Lambda_L} \\ p \succ e}} (Q_p - Q_p^*) Q_e, & G = SO(N), \\ -\frac{1}{2}N\beta \sum_{\substack{p \in \mathcal{P}_{\Lambda_L} \\ p \succ e}} \left((Q_p - Q_p^*) - \frac{1}{N} \text{Tr}(Q_p - Q_p^*) I_N \right) Q_e, & G = SU(N). \end{cases} \quad (6)$$

The Langevin dynamic (5) so constructed has the following standard properties, the details of which can be found in Lemma 3.2 and Lemma 3.3 of [SSZ24].

Proposition 1. *For fixed N and initial data $Q(0) \in \mathcal{Q}_L$, there exists a unique solution $Q \in C([0, \infty), \mathcal{Q}_L)$ to (5). Moreover, the lattice Yang-Mills measure (2) is invariant under (5).*

2.3.2 infinite volume Langevin dynamics

The Langevin dynamic on $\mathcal{Q}$ can also be formally written as in (4), but it's more precisely the SDE system given by (5) at each $e \in E^+$, where L is taken to be ∞ , so that the sum in $\nabla S(Q)_e$ of the form (6) is over all plaquettes in $\mathbb{Z}^d$. Here we explicitly record the infinite volume dynamic:

$$dQ_e = \begin{cases} -\frac{1}{2}N\beta \sum_{\substack{p \in \mathcal{P} \\ p \succ e}} (Q_p - Q_p^*) Q_e dt - \frac{1}{2}(N-1)Q_e dt + \sqrt{2} dB_e Q_e, & \text{if } G = SO(N), \\ -\frac{1}{2}N\beta \sum_{\substack{p \in \mathcal{P} \\ p \succ e}} \left((Q_p - Q_p^*) - \frac{1}{N} \text{Tr}(Q_p - Q_p^*) I_N \right) Q_e dt; -\frac{N^2-1}{N} Q_e dt + \sqrt{2} dB_e Q_e, & \text{if } G = SU(N). \end{cases} \quad (7)$$

In parallel to the finite volume case, the authors are able to show in Proposition 3.4 of [SZZ23] the existence and uniqueness of the solution to (7), after some stochastic analysis acrobatics:

Proposition 2. *For fixed N and initial data $Q(0) \in \mathcal{Q}_L$, there exists a unique strong solution $Q \in C([0, \infty), \mathcal{Q})$ to (7).*

Denote the Markov semigroup associated to the finite volume dynamics by P_t^L , and the semigroup associated to the infinite volume dynamics by P_t .

2.4 finite volume vs. infinite volume

Let us compare Proposition 1 to Proposition 2. We note that in the finite volume case, the lattice Yang-Mills measure is well-defined and easily checked to be invariant under the finite volume dynamic (5). On the other hand, the formula (2) does not make sense in infinite volume, so we seek a deeper understanding of the invariant measure for the infinite volume dynamic (7).

To compare the finite volume invariant measure to the one in infinite volume, we note that each $\mu_{\Lambda_L, N, \beta}$ can be extended to a measure on $\mathcal{Q}$ periodically. The compactness of G and $\mathcal{Q}$ then implies that $\{\mu_{\Lambda_L, N, \beta}\}_{L \geq 1}$ is tight. We have the following observation relating the finite and infinite volume invariant measures:

Theorem 1 (Theorem 3.5 in [SZZ23]). *Every tight limit of $\{\mu_{\Lambda_L, N, \beta}\}_{L \geq 1}$ is an invariant measure for the infinite volume dynamic (7).*

Proof Sketch. Let $\mu_{N, \beta}$ be the weak limit of a subsequence which we still denote by $\mu_{\Lambda_L, N, \beta}$. The idea here is to translate the problem to the solution of the SDEs and then back to the measures. That is, by Propositions 1, we consider unique solutions Q_L to (5) with initial distribution $\mu_{\Lambda_L, N, \beta}$. By verifying a standard upper bound for moments of the form $\mathbb{E}|Q_L(t) - Q_L(s)|^{2p}$, we see that the collection of processes $\{Q_L\}_{L \geq 1}$ meets the Kolmogorov criterion for tightness in $C([0, \infty), \mathcal{Q})$. We can further check that any such tight limit is a solution to the infinite volume dynamic (7) with initial distribution $\mu_{N, \beta}$. By Proposition 2, such a solution is unique and we can denote it by Q . As $\mu_{\Lambda_L, N, \beta}$ is the distribution of $Q_L(t)$ for all $t \geq 0$, taking $L \rightarrow \infty$ we see that $\mu_{N, \beta}$ is the distribution of $Q(t)$ for all $t \geq 0$, so that $\mu_{N, \beta}$ is indeed invariant. $\square$

A priori, the sequence $\{\mu_{\Lambda_L, N, \beta}\}_{L \geq 1}$ may have multiple tight limits. The core theorem of [SZZ23] says that this is not the case in the strong coupling regime. More precisely, we consider small β satisfying

$$K_S := \begin{cases} \frac{N+2}{4} - 1 - 8N|\beta|(d-1) > 0, & G = SO(N), \\ \frac{N+2}{2} - 1 - 8N|\beta|(d-1) > 0, & G = SU(N), \end{cases} \quad (8)$$

Theorem 2 (Theorem 1.2 in [SZZ23]). *Under assumption (8)*

1. *The invariant measure for the Langevin dynamic (7) is unique. Denote this measure by $\mu_{N, \beta}^{\text{YM}}$.*
2. *The sequence $\{\mu_{\Lambda_L, N, \beta}\}_{L \geq 1}$ has a unique tight limit. In fact, $\mu_{\Lambda_L, N, \beta}$ converges weakly to $\mu_{N, \beta}^{\text{YM}}$.*

3. There exists a constant $a > 1$ such that for any probability measure ν on $\mathcal{Q}$,

$$W_2^{\rho_{\infty,a}}(\nu P_t, \mu_{N,\beta}^{YM}) \leq C e^{-\tilde{K}_S t}, \quad (9)$$

where the constant C depends on a and $\tilde{K}_S$ depends on a, d, β, G .

Although stated last, the main content of Theorem 2 is the exponential contraction in Wasserstein distance. We don't have space to cover the details of the proof for 2. Roughly speaking, the strategy is to approximate the infinite volume case by the finite volume case. A suitable coupling is constructed so that the contraction in Lemma 5.1 of [SZZ23] holds. It says that if the explicit constant $\tilde{K}_S$ is positive, then for any initial probability measures μ, ν on $\mathcal{Q}_L$,

$$W_2^{\rho_{\infty,a}}(\mu P_t^L, \nu P_t^L) \leq e^{-\tilde{K}_S t} W_2^{\rho_{\infty,a}}(\mu, \nu).$$

The coupling $(Q(t), Q'(t))_{t \geq 0}$ contributes to the proof of this contraction by showing that for any initial starting points $Q(0)$ and $Q'(0)$,

$$\rho_{\infty,a}^2(Q(t), Q'(t)) \leq e^{-2\tilde{K}_S t} \rho_{\infty,a}^2(Q(0), Q'(0)).$$

The rest of the proof hinges on checking the assumption of Gronwall's lemma for the above inequality to hold, by explicitly working through the geometry involved in the coupling.

3 Functional inequalities

In finite volume, understanding exponential ergodicity does not require sophisticated coupling. Exponential contraction in Wasserstein distance is equivalent to the so-called Bakry-Émery condition, which in turn implies log-Sobolev and Poincaré inequalities. In fact, although the authors of [SZZ23] are able to state the log-Sobolev and Poincaré inequalities for the infinite volume measure thanks to results in Theorem 2, the finite volume functional inequalities discussed in this section often suffice for application to other problems in lattice Yang-Mills theory.

We explain in this section that the functional inequalities in [SZZ23] follow the standard theory around the Bakry-Émery condition. In particular, the constant K_S in (8) is chosen so that the Bakry-Émery condition holds with K_S appearing in the lower bound.

3.1 General facts about functional inequalities

For a finite dimensional Riemannian manifold M with distance ρ , measure $d\mu_M \propto e^S d\text{vol}_M$, and the Langevin diffusion semigroup P_t with generator $\mathcal{L} = \Delta + \nabla S \cdot \nabla$, it is well known the the following conditions are equivalent for any $K \in \mathbb{R}$:

- For all tangent vectors $v \in TM$,

$$\text{Ric}(v, v) - \text{Hess}_S(v, v) \geq K|v|^2 \quad (10)$$

- For all $t \geq 0$ and any probability measures μ, ν on M absolutely continuous with respect to μ_M with finite second moments,

$$W_2^\rho(\mu P_t, \nu P_t) \leq e^{-Kt} W_2^\rho(\mu, \nu) \quad (11)$$

See, for instance, Theorem 9.7.2 in [BGL14]. To better discuss the functional inequalities, we introduce the following notations for $f \in C^\infty(M)$

$$\begin{aligned} \text{Var}_\mu(f) &:= \int f^2 d\mu_M - \left(\int f d\mu_M \right)^2 \\ \text{Ent}_\mu(f) &:= \int f \log f d\mu_M - \left(\int f d\mu_M \right) \log \left(\int f d\mu_M \right) \\ \mathcal{E}(f) &:= \int |\nabla f|^2 d\mu_M \end{aligned} \quad (\text{Dirichlet form})$$

Once we have the Bakry-Émery condition (10) (equivalently (11)), we can obtain the inequalities

$$\text{Ent}_\mu(f^2) \leq \frac{2}{K} \mathcal{E}(f) \quad (\text{Log-Sobolev})$$

$$\text{Var}_\mu(f) \leq \frac{1}{K} \mathcal{E}(f) \quad (\text{Poincare})$$

The details can be found in Propositions 4.8.1 and 5.7.1 of [BGL14].

3.2 Functional inequalities for lattice Yang-Mills

In our setting, the Riemannian manifold in question is $\mathcal{Q}_L$ equipped with the lattice Yang-Mills measure (2). The generator of the Langevin dynamic (5) is $\mathcal{L}_L = \Delta_L + \nabla_L S_{\Lambda_L, N, \beta} \cdot \nabla_L$, where the gradient and Laplacian operators are as in (3) and $S_{\Lambda_L, N, \beta}$ is the lattice Yang-Mills action (1). The Dirichlet form is then explicitly

$$\mathcal{E}_L(f) = \int_{\mathcal{Q}_L} \langle \nabla_L f, \nabla_L f \rangle_{T_Q \mathcal{Q}_L} d\mu_{\Lambda_L, N, \beta} = \sum_{e \in E_{\Lambda_L}^+} \int_{\mathcal{Q}_L} \text{Tr}(\nabla_e f (\nabla_e f)^*) d\mu_{\Lambda_L, N, \beta}.$$

We want to verify the Bakry-Émery condition (10) in this setting. For $G = SO(N)$ or $SU(N)$, the Ricci tensor for any tangent vector u of G is constant, with

$$\text{Ric}(u, u) = \begin{cases} (\frac{N+2}{4} - 1)|u|^2 & G = SO(N) \\ (\frac{N+2}{2} - 1)|u|^2 & G = SU(N) \end{cases}$$

Since the Ricci tensor for a product manifold is additive, we get that for $v \in T_Q \mathcal{Q}_L$,

$$\text{Ric}(v, v) = \begin{cases} (\frac{N+2}{4} - 1)|v|^2 & G = SO(N) \\ (\frac{N+2}{2} - 1)|v|^2 & G = SU(N) \end{cases} \quad (12)$$

An estimate in Lemma 4.1 of [SZZ23] says that

$$|\text{Hess}_S(v, v)| \leq 8(d-1)N|\beta||v|^2 \quad (13)$$

Combining (12) and (13), we see that the Bakry-Émery condition (10) holds in this case with constant K_S defined in (8). When $|\beta|$ is small enough that $K_S > 0$, we therefore obtain the functional inequalities (Log-Sobolev) and (Poincare) with constant $K = K_S$.

4 Application I: Large N Limit of Wilson Loops

As a primary application of the functional inequalities established in the previous section, we examine the behavior of Wilson loops in the large N limit. In the physics literature, it is expected that in the limit $N \rightarrow \infty$ (with 't Hooft scaling $\beta \sim 1/N$), the expectations of gauge-invariant observables factorize. While this has been explored via Makeenko-Migdal loop equations on finite lattices, the stochastic analysis framework of [SZZ23] allows for a direct proof in infinite volume using the Poincaré inequality (Poincare).

4.1 Variance bound via Poincaré inequality

Consider a closed loop l consisting of a sequence of edges $e_1, \dots, e_n$. The Wilson loop observable is defined as the trace of the holonomy along l :

$$W_l(Q) := \text{Tr}(Q_{e_1} \cdots Q_{e_n}).$$

We are interested in the normalized observable $f_l(Q) := \frac{1}{N} W_l(Q)$. A central result of [SZZ23] (Corollary 1.5) is a quantitative bound on the variance of this observable under the infinite volume measure $\mu_{N,\beta}^{\text{YM}}$.

Since we have established the Poincaré inequality (Poincare) with constant K_S , we have the bound:

$$\text{Var}_{\mu_{N,\beta}^{\text{YM}}} \left(\frac{W_l}{N} \right) \leq \frac{1}{K_S} \mathcal{E} \left(\frac{W_l}{N} \right) = \frac{1}{K_S N^2} \sum_{e \in E^+} \int |\nabla_e W_l|^2 d\mu_{N,\beta}^{\text{YM}}. \quad (14)$$

To utilize this, one must estimate the energy of the Wilson loop. The gradient $\nabla_e W_l$ is non-zero only for edges e contained in the loop l . Explicit computations using the Lie algebra basis (see Section 4.2 of [SZZ23]) reveal that the magnitude of the gradient $\nabla_e W_l$ scales naturally with N . Specifically, summing over the edges in the loop yields a bound on the Dirichlet form that is independent of the state Q :

$$\mathcal{E} \left(\frac{W_l}{N} \right) \leq \frac{C_n}{N},$$

where C_n depends on the length of the loop n but not on the matrix dimension N .

Substituting this energy estimate into (14), we obtain the specific bounds derived in [SZZ23]:

$$\text{Var}_{\mu_{N,\beta}^{\text{YM}}} \left(\frac{W_l}{N} \right) \leq \begin{cases} \frac{n(n-3)}{K_S N} & G = SO(N), \\ \frac{4n(n-3)}{K_S N} & G = SU(N). \end{cases} \quad (15)$$

Crucially, since the curvature lower bound K_S defined in (8) is positive and uniform in N (under the 't Hooft scaling assumption where βN is kept constant or small), the right-hand side of (15) vanishes as $N \rightarrow \infty$.

4.2 Factorization

The variance bound (15) has immediate physical consequences for the large N theory. First, it implies that the normalized Wilson loop $\frac{1}{N}W_l$ converges in probability to its deterministic expectation as $N \rightarrow \infty$. Second, it implies the factorization property for multi-loop observables. For any two loops l_1 and l_2 , the Cauchy-Schwarz inequality yields:

$$\begin{aligned} \left| \mathbb{E} \left[\frac{W_{l_1}}{N} \frac{W_{l_2}}{N} \right] - \mathbb{E} \left[\frac{W_{l_1}}{N} \right] \mathbb{E} \left[\frac{W_{l_2}}{N} \right] \right| &= \left| \text{Cov} \left(\frac{W_{l_1}}{N}, \frac{W_{l_2}}{N} \right) \right| \\ &\leq \sqrt{\text{Var} \left(\frac{W_{l_1}}{N} \right) \text{Var} \left(\frac{W_{l_2}}{N} \right)} \\ &\leq O \left(\frac{1}{N} \right). \end{aligned}$$

Thus, in the large N limit, the expectation of a product of Wilson loops factorizes into the product of their expectations:

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mu_{N,\beta}^{\text{YM}}} \left[\prod_{i=1}^k \frac{W_{l_i}}{N} \right] = \prod_{i=1}^k \lim_{N \rightarrow \infty} \mathbb{E}_{\mu_{N,\beta}^{\text{YM}}} \left[\frac{W_{l_i}}{N} \right].$$

This result, often assumed as a premise in the derivation of the Master Loop Equation in physics literature, is here rigorously proven as a consequence of the concentration of measure phenomenon induced by the high-dimensional geometry of the gauge group.

Application II: Dynamical derivation of Wilson's area law

A second major application of the stochastic mass gap is a rigorous derivation of Wilson's area law in the strong coupling regime. While the mass gap (exponential decay of correlations) discussed in Application I suggests confinement, proving the precise area law bound

$$|\mathbb{E}[W_l]| \leq C e^{-c \cdot \text{Area}(l)}$$

requires a more specialized argument to handle the geometry of large loops. Recently, [CNS25] showed that the Langevin–Bakry–Émery framework can be used to verify the classical “mass gap conditio” of [DF80].

The strategy relies on a layer-by-layer reduction argument. Consider a rectangular loop l of size $L \times T$ lying in the (x_1, x_2) plane. We decompose the lattice into a sequence of “slabs” $S_k = \{e \in E_\Lambda^+ : k \leq x_2(e) \leq k+1\}$. If we fix the gauge field configuration ξ on the boundary ∂S_k (the hyperplanes at $x_2 = k$ and $x_2 = k+1$), the conditional distribution of the edges within the slab, denoted $\mu_{S_k}^\xi$, is governed by a conditional action $S_{S_k}^\xi$. Crucially, the Hessian of this conditional action inherits the convexity properties of the global action. cao2025dynamical prove that for sufficiently small β , the Bakry–Émery curvature of the conditional measure satisfies a uniform lower bound:

$$\text{Ric}_{S_k}^\xi \geq \kappa > 0, \quad \text{uniformly in } \xi.$$

This uniform curvature bound implies a uniform Poincaré inequality for the conditional measures. Specifically, for any observable F supported on the slab S_k :

$$\text{Var}_{\mu_{S_k}^\epsilon}(F) \leq \frac{1}{\kappa} \mathbb{E}_{\mu_{S_k}^\epsilon}[|\nabla F|^2].$$

This allows one to verify the Durhuus–Fröhlich condition, which essentially states that the correlation between the “top” and “bottom” segments of the Wilson loop decays exponentially across the slab. Let $W_{L \times T}$ be the Wilson loop variable for a rectangle of height T . We can view this as the correlation of two line operators separated by distance T . Applying the uniform spectral gap to the conditional expectation on the k -th slab, one derives a recursive bound:

$$|\mathbb{E}[W_{L \times T}]| = |\mathbb{E}[\mathbb{E}[W_{L \times T} | \mathcal{F}_{\partial S_{T-1}}]]| \leq \epsilon(L) \cdot |\mathbb{E}[W_{L \times (T-1)}]|,$$

where $\epsilon(L) \approx e^{-cL}$ represents the contraction of the operator norm across one slab layer. Iterating this procedure T times yields the area law:

$$|\mathbb{E}[W_{L \times T}]| \leq (\epsilon(L))^T \approx e^{-cLT} = e^{-c \cdot \text{Area}(l)}.$$

Thus, the stochastic analysis approach provides the microscopic justification for the stability of the slab measures. The Riemannian geometric stability of the Langevin dynamics serves as the engine that drives the macroscopic confinement property, allowing a fully dynamical derivation of the area law for gauge groups $U(N)$, $SU(N)$, and $SO(N)$ in the ‘t Hooft regime.

5 Conclusion

This report has reviewed the stochastic analysis approach to lattice Yang-Mills theory, primarily following the framework established in [SZZ23]. By replacing the traditional combinatorial techniques of cluster expansions with the tools of stochastic quantization, this approach offers a robust geometric perspective on gauge theories in the strong coupling regime.

The core achievement of this framework is the rigorous construction of the infinite-volume lattice Yang-Mills measure $\mu_{N,\beta}^{\text{YM}}$ as the unique invariant measure of a Langevin dynamic on the configuration space $G^{\mathbb{Z}^d}$. We detailed how this uniqueness is established not through static bounds on the partition function, but through a dynamical coupling argument involving a weighted Wasserstein distance. This method explicitly utilizes the Riemannian geometry of the compact Lie groups $SO(N)$ and $SU(N)$, showing that ergodicity is driven by the positive curvature of the group manifold dominating the fluctuations of the action.

Furthermore, we demonstrated that the strong coupling assumption ($K_S > 0$) is equivalent to the Bakry–Émery curvature-dimension condition. This yields powerful functional inequalities—specifically the Log-Sobolev and Poincaré inequalities—directly in the infinite volume limit. These inequalities serve as the mathematical engine for deriving key physical properties of the theory:

- **Large N Factorization:** The Poincaré inequality provides a variance bound scaling as $O(1/N)$, proving that Wilson loops concentrate around their expectations and factorize in the large N limit.

- **Mass Gap and Confinement:** The same geometric estimates imply the exponential decay of correlations (mass gap) and, via a layer-by-layer reduction, the celebrated area law for Wilson loops.

In summary, the Langevin dynamic approach unifies the analytic, geometric, and probabilistic aspects of lattice gauge theory. It provides a distinct and transparent pathway to understanding the stability of the vacuum and the confinement of observables, validating 't Hooft's predictions for large N gauge theories with rigorous finite- N estimates.

References

- [BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and Geometry of Markov Diffusion Operators*. Vol. 348. Grundlehren der mathematischen Wissenschaften. Springer, 2014.
- [Cha17] Sourav Chatterjee. “Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit”. In: (2017). URL: <https://arxiv.org/abs/1502.07719>.
- [CNS25] Sky Cao, Ron Nissim, and Scott Sheffield. “Dynamical approach to area law for lattice Yang-Mills”. In: *arXiv preprint arXiv:2509.04688* (2025).
- [DF80] B Durhuus and J Fröhlich. “A connection between ν -dimensional Yang-Mills theory and $(\nu-1)$ -dimensional, non-linear σ -models”. In: *Communications in Mathematical Physics* 75.2 (1980), pp. 103–151.
- [SSZ24] Hao Shen, Scott A Smith, and Rongchan Zhu. “A new derivation of the finite N master loop equation for lattice Yang-Mills”. In: *Electronic Journal of Probability* 29 (2024), pp. 1–18.
- [SZZ23] Hao Shen, Rongchan Zhu, and Xiangchan Zhu. “A stochastic analysis approach to lattice Yang-Mills at strong coupling”. In: *Communications in Mathematical Physics* 400.2 (2023), pp. 805–851.
- [Wil74] Kenneth G. Wilson. “Confinement of quarks”. In: *Phys. Rev. D* 10 (8 Oct. 1974), pp. 2445–2459. URL: <https://link.aps.org/doi/10.1103/PhysRevD.10.2445>.