

Random surfaces and Lattice Yang-Mills—A Synopsis

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Abstract

This short report aims to flesh out the main ideas in [1]. For more context and proof details, the reader is encouraged to consult the original paper and the references therein.

1 Wilson loop expectations as a combinatorial sum

1.1 Taylor expanding Wilson loop expectations

At a heuristic level, the Euclidean Yang-Mills theory studies a random ensemble of Lie-algebra-valued connections on some Riemannian manifold. Discretizing with a finite lattice $\Lambda \subset \mathbb{Z}^d$, this random ensemble can be formalized as a measure on configurations of elements in $U(N)$ attached to each edge in Λ , where $U(N)$ is the group of $N \times N$ unitary matrices.

We shall fix some notations to define this measure. It is conventional to call a square in the lattice Λ a **plaquette**. We endow each edge and plaquette in Λ with an orientation. Denote by E^+ the set of positively oriented edges, and $\mathcal{P}$ the set of positively and negatively oriented plaquettes, so that each plaquette appears in the collection $\mathcal{P}$ twice with opposite orientations. We associate a unitary U_e to each $e \in E^+$, and set $U_{e^{-1}} := (U_e)^{-1}$. For each $p \in \mathcal{P}$, let U_p denote the product of the four unitaries attached to the four edges of p respecting orientations, as in Figure 1. Let dU_e denote the Haar measure on the copy of $U(N)$ attached to $e \in E^+$. Then the lattice Yang-Mills measure $\mu_{\Lambda, \beta}$ has density with respect to the product Haar measure given

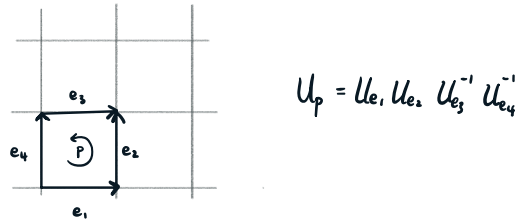


Figure 1: product of unitaries that appears in the lattice Yang-Mills measure

by local plaquette couplings:

$$d\mu_{\Lambda,\beta}(U) = Z_{\Lambda,\beta}^{-1} \prod_{p \in \mathcal{P}} \exp(N\beta \text{Tr}(U_p)) \prod_{e \in E^+} dU_e$$

One particular quantity of interest is the expectation of the so-called **Wilson loop** observables under this measure. Given a collection of oriented loops $s = (s_1, \dots, s_J)$ in Λ , set $U_{s_j} := \prod_{e \in s_j} U_e$. Let $\text{tr}(\cdot)$ denote normalized trace. Then the expectation we wish to calculate is

$$\begin{aligned} \langle \prod_{j \in [J]} \text{tr}(U_{s_j}) \rangle_{\Lambda,\beta} &= Z_{\Lambda,\beta}^{-1} \int \prod_{j \in [J]} \text{tr}(U_{s_j}) \prod_{p \in \mathcal{P}} \exp(N\beta \text{Tr}(U_p)) \prod_{e \in E^+} dU_e \\ &= Z_{\Lambda,\beta}^{-1} \sum_{K: \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}} \frac{(N\beta)^K}{K!} \int \prod_{j \in [J]} \text{tr}(U_{s_j}) \prod_{p \in \mathcal{P}} \text{Tr}(U_p)^{K(p)} \prod_{e \in E^+} dU_e \end{aligned} \quad (1.1)$$

where we taylor expanded each exponential term to get the second line. If we fix $K: \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$, then the problem reduces to the calculation of a Haar integral.

1.2 Haar integral as diagrammatic sum

It is helpful to reformulate the Haar integral in a more abstract way without reference to the lattice Λ , so as to facilitate combinatorics. Since we are fixing a particular K , in this subsection we work with unitaries distributed according to the product Haar measure. Let $|E^+| = L$. Consider the following way to index products of Haar unitaries:

- letters $\{\lambda_1, \dots, \lambda_L\} \longleftrightarrow$ independent Haar unitaries $\{U_{\lambda_1}, \dots, U_{\lambda_L}\}$ attached to $\lambda_i \in E^+$
- words $\Gamma = (\Gamma_1, \dots, \Gamma_k) \longleftrightarrow$ products of Haar unitaries whose traces appear in (1.1)

In more detail, we may write

$$\Gamma_i = \lambda_{c_i(1)}^{\varepsilon_i(1)} \dots \lambda_{c_i(m_i)}^{\varepsilon_i(m_i)} \longleftrightarrow U(\Gamma_i) = U_{\lambda_{c_i(1)}}^{\varepsilon_i(1)} \dots U_{\lambda_{c_i(m_i)}}^{\varepsilon_i(m_i)}$$

where $c_i: [m_i] \rightarrow [L]$ encodes the ordering of each unitary appearing in the product $U(\Gamma_i)$ of length m_i , and $\varepsilon_i: [m_i] \rightarrow \{\pm 1\}$ keeps track of the orientation of each unitary. Applying this notation to (1.1), one may think of Γ_i as either a word of length 4 corresponding to $U(\Gamma_i) = U_p$ for a plaquette p , or a possibly longer word corresponding to $U(\Gamma_i) = U_{s_i}$ for some s_i in the collection of Wilson loops. Up to factors of N resulting from normalized trace, computing a summand in (1.1) for a fixed K amounts to calculating the Haar integral

$$\mathbb{E}[\text{Tr}(U(\Gamma))] := \mathbb{E}_{\prod_i \text{Haar}_i} [\text{Tr}(U(\Gamma_1)) \dots \text{Tr}(U(\Gamma_k))] \quad (1.2)$$

The calculation of such unitary Haar integrals is well-understood with the use of Weingarten calculus. For this purpose, we need to consider **Weingarten functions** $\text{Wg}_N: \cup_n S_n \rightarrow \mathbb{Q}$ with the defining property that

$$\mathbb{E}_{\text{Haar}} [U_{i_1 j_1} \dots U_{i_n j_n} \bar{U}_{i'_1 j'_1} \dots \bar{U}_{i'_n j'_n}] = \sum_{\sigma, \tau \in S_n} \underbrace{\delta_{i_1 i'_{\sigma(1)}} \dots \delta_{i_n i'_{\sigma(n)}}}_{\delta_{\bar{i}\sigma(\bar{i}')}} \underbrace{\delta_{j_1 j'_{\tau(1)}} \dots \delta_{j_n j'_{\tau(n)}}}_{\delta_{j\tau(\bar{j}')}} \text{Wg}_N(\sigma\tau^{-1}) \quad (1.3)$$

Note that by symmetries of the Haar measure, the Haar integrals of other products of matrix entries vanish, so the Weingarten function completely encodes the moments of a Haar unitary. Since product of traces can be expanded as a sum of products of matrix entries, the integral (1.2) can be directly computed after applying (1.3). One main point in [1] is that this computation can be interpreted as a sum over certain diagrams. Due to the aforementioned vanishing of Haar integrals, we only need to consider diagrams arising from **balanced** collection of words, meaning that each λ_l appears n_l times as λ_l^{+1} and the same n_l times as λ_l^{-1} .

Definition 1.4 (Strand Diagrams I). *Given a balanced collection of words $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ in letters $\{\lambda_1, \dots, \lambda_L\}$ and a pair of permutations for each $l \in [L]$ collected in $\pi = ((\sigma_1, \tau_1), \dots, (\sigma_L, \tau_L))$, $\sigma_l, \tau_l \in S_{n_l}$, a **strand diagram** is formed with the following ingredients:*

- Collect all the letters appearing in $\Gamma_1, \dots, \Gamma_k$ into columns indexed by λ_l . In the column of λ_l , draw n_l rightward pointing strands and n_l leftward pointing strands
- Draw **red links/exterior connections** between the strands corresponding to the words Γ_i
- Draw **blue links/interior connections** according to σ_l, τ_l , viewing σ_l as a matching on the left side of the column of λ_l and τ_l as a matching on the right side.

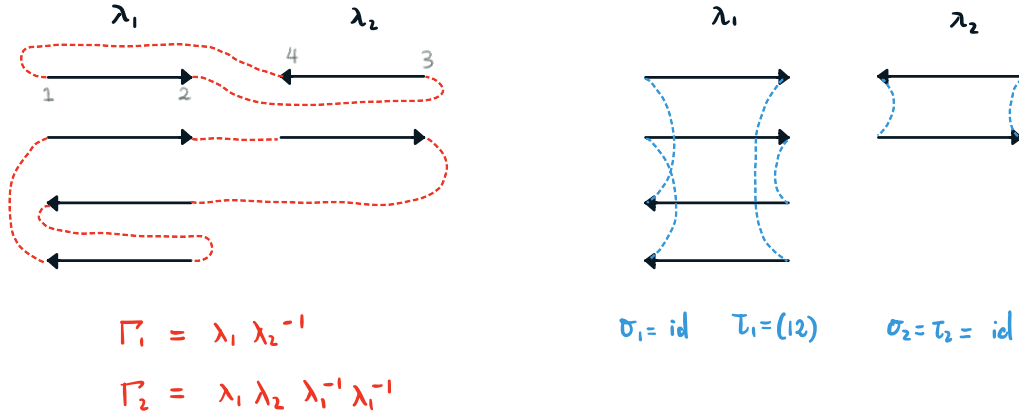


Figure 2: Example of a strand diagram. The red links and blue links should be drawn together on the same diagram, but they are separated here for visual clarity.

This definition is best illustrated with an example: see Figure 2. Given a strand diagram specified by Γ and π , we may forget the black strands to get a graph where the vertices are the $4 \sum_l n_l$ ends of the strands and the edges are the red and blue links. Let $\# \text{comp}(\Gamma, \pi)$ be the number of components of this graph: see Figure 3. With the above definitions, we are ready to state one main result in [1].

Theorem 1.5 (Theorem 2.5 in [1]). *For a balanced collection of words $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ on letters $\{\lambda_1, \dots, \lambda_L\}$,*

$$\mathbb{E}[\text{Tr}(U(\Gamma))] = \sum_{\pi = (\sigma_l, \tau_l \in S_{n_l}, l \in [L])} \left(\prod_{l \in [L]} W_{g_N}(\sigma_l \tau_l^{-1}) \right) N^{\# \text{comp}(\Gamma, \pi)}.$$

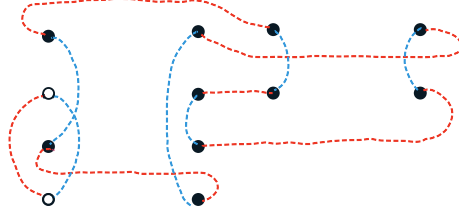


Figure 3: The same strand diagram as in Figure 2 with strands removed. Note that in this case $\#\text{comp}(\Gamma, \pi) = 2$

As remarked above, this computation is well-known, but the interpretation in terms of strand diagrams is particularly clean and help connect this Haar integral to the analogous expectation of unitary Brownian motion studied in a companion paper [3]. The following proof is not spelled out in [1], but the authors mention that it could be readily done. We follow the lectures by Sky Cao [2].

Proof of Theorem 1.5. First, we observe that red links in a strand diagram keep track of the indices of matrix entries when taking the trace of a product of unitaries. For instance, consider $\Gamma_1 = \lambda_1 \lambda_2^{-1}$ as in Figure 2. Using Einstein summation convention, we may write $\text{Tr}(U(\Gamma_1)) = (U_{\lambda_1})_{i_1 i_2} \delta^{i_2 i_3} (U_{\lambda_2}^{-1})_{i_3 i_4} \delta^{i_4 i_1} = (U_{\lambda_1})_{i_1 i_2} \delta^{i_2 i_3} (\bar{U}_{\lambda_2})_{i_4 i_3} \delta^{i_4 i_1}$.

More generally, we introduce the following notations. Let $R(\Gamma)$ be the set of red links in the strand diagram. Divide the black strands in the strand diagrams into the positively oriented collection $B_+(\Gamma)$ and the negatively oriented collection $B_-(\Gamma)$. For each strand e , let $\lambda(e)$ be the letter that e corresponds to. Moreover, label the vertices in the strand diagram arbitrarily, and let i_e (resp. i'_e) denote the the left vertex of the positively (resp. negatively) oriented strand e . Similarly, let j_e or j'_e denote the right vertex of the strand e , depending on the orientation of e . We also label the vertices of a red link e as k_e and l_e . We have

$$\text{Tr}(U(\Gamma)) := \text{Tr}(U(\Gamma_1)) \cdots \text{Tr}(U(\Gamma_k)) = \prod_{e \in R(\Gamma)} \delta^{k_e l_e} \prod_{e \in B_+(\Gamma)} (U_{\lambda(e)})_{i_e j_e} \prod_{e \in B_-(\Gamma)} (\bar{U}_{\lambda(e)})_{i'_e j'_e}$$

When taking the Haar expectation, the expectation of product factors as a product of expectations, by the independence of $\{U_{\lambda_l} : l \in [L]\}$. We may then apply the formula (1.3) to get

$$\begin{aligned} \mathbb{E}[\text{Tr}(U(\Gamma))] &= \prod_{e \in R(\Gamma)} \delta^{k_e l_e} \mathbb{E}_{\Pi_l, \text{Haar}_l} \left[\prod_{e \in B_+(\Gamma)} (U_{\lambda(e)})_{i_e j_e} \prod_{e \in B_-(\Gamma)} (\bar{U}_{\lambda(e)})_{i'_e j'_e} \right] \\ &= \prod_{e \in R(\Gamma)} \delta^{k_e l_e} \prod_{l \in [L]} \mathbb{E}_{\text{Haar}_l} \left[\prod_{e \in B_+(\Gamma), \lambda(e)=\lambda_l} (U_{\lambda(e)})_{i_e j_e} \prod_{e \in B_-(\Gamma), \lambda(e)=\lambda_l} (\bar{U}_{\lambda(e)})_{i'_e j'_e} \right] \\ &= \sum_{\pi=(\sigma_l, \tau_l \in S_{n_l}, l \in [L])} \prod_{e \in R(\Gamma)} \delta^{k_e l_e} \prod_{l \in [L]} \delta_{i_l \sigma_l(\bar{i}'_l)} \delta_{j_l \tau_l(\bar{j}'_l)} \text{Wg}_N(\sigma_l \tau_l^{-1}) \\ &= \sum_{\pi=(\sigma_l, \tau_l \in S_{n_l}, l \in [L])} \left(\prod_{l \in [L]} \text{Wg}_N(\sigma_l \tau_l^{-1}) \right) N^{\#\text{comp}(\Gamma, \pi)}. \end{aligned}$$

We see that the interpretation of σ_l and τ_l as left and right matchings in strand diagrams is consistent with the matching of indices in Weingarten calculus. The last equality follows because the number of free variables in the sum over upper and lower indices is the number of components in the strand diagram specified by Γ and π .

□

2 Wilson loop expectations as a sum over surfaces

Applying Theorem 1.5 to the expansion (1.1) yields an expression of Wilson loop expectations as a double sum over plaquette weights $K : \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$ and pairs of bijections $\pi = (\sigma_l, \tau_l \in S_{n_l}, l \in [L])$. In section 3 of [1], the authors interpret the resulting sum as a sum over surfaces, hence the title of the paper. This is done in two steps.

2.1 From diagrams to maps

A strand diagram can naturally be interpreted as a planar map. The vertex set is given by the ends of black strands modulo equivalence by red and blue links. Black strands form the edges. Each Γ_i gives rise to a red face, whereas each pair of left and right bijections $\sigma_l, \tau_l \in S_{n_l}$ correspond to blue faces formed from strands of λ_l . Note that each edge is incident to exactly one red face and one blue face, by the construction of strand diagrams. See Figure 4.

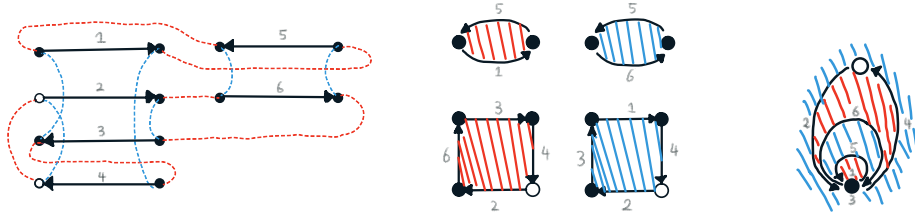


Figure 4: The same strand diagram as in Figure 2. Red and blue links correspond to red and blue faces. The two vertices are drawn as solid and hollow circles. Combining the faces gives rise to a planar map whose faces are bipartite.

The sum over $\sigma_l, \tau_l \in S_{n_l}$ can thus be expressed in terms of dual-bipartite maps.

Definitions 2.1. Given a balanced collection of words Γ , let $\text{DBM}(\Gamma)$ be the set of maps corresponding to strand diagrams specified by Γ and additional pairs of left and right bijections. For a map $\mathcal{M} \in \text{DBM}(\Gamma)$, let $\mu_l(\mathcal{M})$ be the partition of n_l given by $1/2$ times the degrees of blue faces associated to λ_l .

The introduction of the partitions $\mu_l(\mathcal{M})$ allows us to write the Weingarten functions appearing in 1.5 in terms of maps. We may read $\sigma_l \tau_l^{-1}$ as tracing blue links from left to right in the l -th column of the strand diagram. Since the value of the Weingarten function only depends on the cycle type of the permutation, we get $\text{Wg}_N(\sigma_l \tau_l^{-1}) = \text{Wg}_N(\mu_l(\mathcal{M}))$.

In view of the formulas to be obtained, it is more natural to work with the normalized Weingarten function $\overline{\text{Wg}}_N$, which has a nontrivial large N limit. Let the Euler characteristic

$\chi(\mathcal{M})$ of a dual-bipartite map $\mathcal{M}$ be the usual $\# \text{vertices} - \# \text{edges} + \# \text{faces}$. After some counting, the following reformulation of Theorem 1.5 is obtained.

Proposition 2.2 (Proposition 3.8 in [1]). *For a balanced collection of words $\Gamma = (\Gamma_1, \dots, \Gamma_k)$ on letters $\{\lambda_1, \dots, \lambda_L\}$,*

$$\mathbb{E}[\text{Tr}(U(\Gamma))] = \sum_{\mathcal{M} \in \text{DBM}(\Gamma)} \left(\prod_{l \in [L]} \overline{Wg}_N(\mu_l(\mathcal{M})) \right) N^{\chi(\mathcal{M})-k}.$$

In (1.1), the words $\Gamma = \Gamma(s, K)$ are determined by the Wilson loops $s = (s_1, \dots, s_J)$ and plaquette weights K . A direct substitution yields

$$\left\langle \prod_{j \in [J]} \text{tr}(U_{s_j}) \right\rangle_{\Lambda, \beta} = Z_{\Lambda, \beta}^{-1} \sum_{K: \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}} \frac{(N\beta)^K}{K!} \sum_{\mathcal{M} \in \text{DBM}(\Gamma(s, K))} \left(\prod_{l \in [L]} \overline{Wg}_N(\mu_l(\mathcal{M})) \right) N^{\chi(\mathcal{M})-k}. \quad (2.3)$$

2.2 From maps to maps with boundary

We have interpreted the expansion (1.1) fixing K and hence $\Gamma(s, K)$, successively as a sum over diagrams and as a sum over maps determined by $\Gamma(s, K)$ and additional permutations. We shall now consider varying plaquette weights K and work with strand diagrams/maps with the underlying lattice in mind. This motivates the following definition:

Definition 2.4. *A pair $(\mathcal{M}, \phi)$ with $\mathcal{M}$ a map and $\phi : \mathcal{M} \rightarrow \Lambda$ a graph homomorphism is an **edge-plaquette embedding** if the dual graph of $\mathcal{M}$ is bipartite with the following properties. One partite class of **blue** faces are each mapped to an **edge** in Λ . The other partite class of **yellow** faces are each mapped onto a **plaquette** in $\mathcal{P}$.*

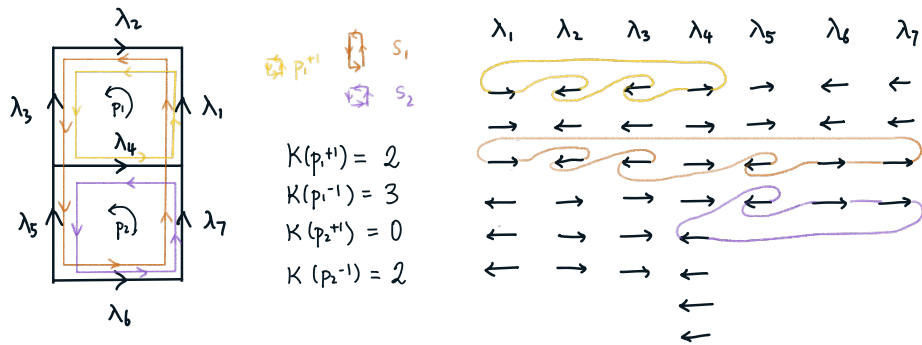


Figure 5: Example of a balanced pair (s, K) . Blue faces and some yellow faces corresponding to plaquettes are not drawn. Note that in the language of strand diagrams, all the links/faces drawn are red links/faces. But in the language of edge-plaquette embeddings, only the faces corresponding to plaquettes are yellow faces. The orange and purple loops are considered as the boundaries of the map.

A map $\mathcal{M} \in \text{DBM}(\Gamma(s, K))$ is almost an edge-plaquette embedding, except that the red faces associated to Wilson loops s might not correspond to plaquettes in Λ . We may remove these faces and obtain an edge-plaquette embedding $(\overline{\mathcal{M}}, \phi)$, where ϕ maps a yellow face in $\overline{\mathcal{M}}$ to the corresponding plaquette and maps a blue face to an edge in Λ .

Let $\text{EPE}(s, K)$ be the set of edge-plaquette embeddings obtained in this way from $\mathcal{M} \in \text{DBM}(\Gamma(s, K))$, and $\text{EPE}(s)$ the disjoint union of $\text{EPE}(s, K)$ over plaquette weights $K : \mathcal{P} \rightarrow \mathbb{Z}_{\geq 0}$. $\text{EPE}(s)$ may be interpreted as the set of all edge-plaquette embeddings with boundary s . Thus, we write $\partial(\overline{\mathcal{M}}, \phi) = s$ to mean $(\overline{\mathcal{M}}, \phi) \in \text{EPE}(s)$. $\phi^{-1}(e)$ encodes the blue faces of λ_l , where λ_l is the letter corresponding to the edge e . We may write the partition $\mu_l(\mathcal{M})$ of n_l in our previous notations as $\mu_e(\phi)$. More suggestively, for $(\overline{\mathcal{M}}, \phi) \in \text{EPE}(s)$, write $\text{area}(\overline{\mathcal{M}}, \phi) = \sum_p K(p)$ and $(\phi^{-1})! = K!$. Applying these notations to (2.3) and performing a simple counting of faces, we are able to absorb the sum over K and express the expectation for a collection of Wilson loops s as in Theorem 3.10 of [1]:

Theorem 2.5. *For a collection of loops $s = (s_1, \dots, s_J)$ in Λ ,*

$$\left\langle \prod_{j \in [J]} \text{tr}(U_{s_j}) \right\rangle_{\Lambda, \beta} = Z_{\Lambda, \beta}^{-1} \sum_{\partial(\overline{\mathcal{M}}, \phi) = s} \frac{\beta^{\text{area}(\overline{\mathcal{M}}, \phi)}}{(\phi^{-1})!} \left(\prod_{e \in E_\Lambda} \overline{Wg}_N(\mu_e(\phi)) \right) N^{\chi(\overline{\mathcal{M}})}. \quad (2.6)$$

3 Poisson point process and unitary Brownian motion

The parts of the paper [1] we have discussed so far focus on the interpretation of Haar integrals appearing in the computation of Wilson loop expectations, based on the result of Theorem 1.5. These interpretations are nice in themselves and have the potential to facilitate future studies. The main technical contribution of the paper is centered on an alternative proof of Theorem 1.5. The authors show that if we associate to the letters $\{\lambda_1, \dots, \lambda_L\}$ independent unitary Brownian motions and use the same notations as in the case of Haar random unitaries, then we may recover $\mathbb{E}[\text{Tr}(U(\Gamma))]$ in Theorem 1.5 as the large T limit of $\mathbb{E}[\text{Tr}(B^T(\Gamma))]$. In this section we try to explain this result and briefly sketch the key ingredients in the proof.

The companion paper [3] derives an expression for expectations of the form $\mathbb{E}[\text{Tr}(B^T(\Gamma))]$ in terms of Poisson point processes on strand diagrams.

Definition 3.1 (Strand Diagrams II). *In the previous definition 1.4 a strand diagram is specified by a balanced collection of words Γ and pairs of left and right bijections $\pi = (\sigma_l, \tau_l \in S_{n_l})$. We may relax the constraints and consider strand diagrams specified by (Γ, π) where*

- Γ is a collection of words on letters $\{\lambda_1, \dots, \lambda_L\}$ that need not be balanced. Let n_l^+ be the number of occurrences of λ_l^+ , and similarly let n_l^- be the number of occurrences of λ_l^- . Write $n_l^\pm := n_l^+ + n_l^-$. Note that this differs from the previous notation n_l by a factor of 2.
- $\pi = (\pi_1, \dots, \pi_L)$ where each π_l is a matching on $2n_l^\pm$.

A matching π_l may be specified by a collection of points as follows. Think of the strands as having the same length $[0, T]$. For each pair of strands in column l we have a collection of points on $[0, T]$. Formally, consider $P \subset \mathcal{D}_l^T$, where $\mathcal{D}_l^T = \bigsqcup_{i, j \in [n_l^\pm], i < j} [0, T]$. We interpret a point $p \in P$ with index $\{i, j\}$ as a line connecting strand i and strand j at the location specified

by p . If i and j have the same orientation, insert a **green** line interpreted as a **swap**. If they have opposite orientations, insert a **blue** line interpreted as a **turnaround**. By tracing the strands, we obtain a matching on $2n_l^\pm$. See Figure 6.

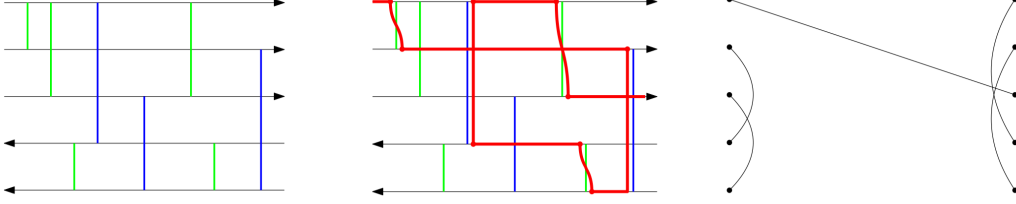


Figure 6: Example of a collection of points in $\mathcal{D}_l^T$ corresponding to a matching on $2n_l^\pm$ (Figures 11-13 of [1]).

We introduce randomness into this correspondence from points to matchings.

Definition 3.2. Given a collection of words Γ on $\{\lambda_1, \dots, \lambda_L\}$, the **Poisson point process** associated to Γ is the process $\Sigma^T = \bigsqcup_{l \in [L]} \Sigma_l^T$, where each Σ_l^T is a rate-1 Poisson point process on D_l^T .

We are now ready to state the expression of unitary Brownian motion expectations in terms of Poisson point processes on strand diagrams. We saw that Σ_l^T correspond to a random matching on $2n_l^\pm$, but it also contains more information. What is relevant here is that we may count the number of interior loops in the column of the strand diagram decorated by Σ_l^T . Let M_n be the set of matchings on $2n$. For a collection of matchings $\pi = (\pi_l \in M_{n_l^\pm}, l \in [L])$, let

$$w^T(\pi_l) := e^{(\frac{n_l^\pm}{2})T - \frac{n_l^\pm}{2}T} \mathbb{E} \left[(-1)^{\#\text{swaps}(\Sigma_l^T)} N^{-|\Sigma_l^T|} N^{\#\text{loops}(\Sigma_l^T)} \mathbb{1}(\pi(\Sigma_l^T) = \pi_l) \right]. \quad (3.3)$$

Theorem 3.4 (Lemma 2.14 in [1]). Let Γ be a collection of words on $\{\lambda_1, \dots, \lambda_L\}$ with the corresponding Poisson point process Σ^T . Then

$$\mathbb{E} \left[\text{Tr}(B^T(\Gamma)) \right] = \sum_{\pi = (\pi_l \in M_{n_l^\pm}, l \in [L])} \left(\prod_{l \in [L]} w^T(\pi_l) \right) N^{\#\text{comp}(\Gamma, \pi)}.$$

This theorem is proven in [3]. In [1] the authors prove that $\lim_{T \rightarrow \infty} \mathbb{E} \left[\text{Tr}(B^T(\Gamma)) \right] = \mathbb{E} \left[\text{Tr}(U(\Gamma)) \right]$ using Theorem 1.5 and Theorem 3.4.

Theorem 3.5. Let Γ be a balanced collection of words on $\{\lambda_1, \dots, \lambda_L\}$. Then

$$\lim_{T \rightarrow \infty} \sum_{\pi = (\pi_l \in M_{2n_l}, l \in [L])} \left(\prod_{l \in [L]} w^T(\pi_l) \right) N^{\#\text{comp}(\Gamma, \pi)} = \sum_{\pi = (\sigma_l, \tau_l \in S_{n_l}, l \in [L])} \left(\prod_{l \in [L]} Wg_N(\sigma_l \tau_l^{-1}) \right) N^{\#\text{comp}(\Gamma, \pi)}.$$

For $\sigma, \tau \in S_n$, let $[\sigma \tau^{-1}]$ be the corresponding matching in M_{2n} . Comparing two sides of the above equation, we observe that it suffices to focus on a single letter λ and prove for $\pi \in M_{2n}$

$$\lim_{T \rightarrow \infty} w^T(\pi) = \sum_{\sigma, \tau \in S_n: \pi = [\sigma \tau^{-1}]} Wg_N(\sigma \tau^{-1}). \quad (3.6)$$

To do so, we need to understand the coefficients $w^T(\pi)$. We shall drop all subscripts l and work in the single letter case. Since the right hand side of (3.3) involves factors that do not only depend on the matching induced by Σ^T , the correct way to understand Σ^T is as random elements in the Brauer algebra.

The **Brauer algebra** $\mathcal{B}_n$ is the group algebra of M_n , where the group operation is given by concatenation of matchings, with an extra factor N for each loop incurred. See Figure 7.

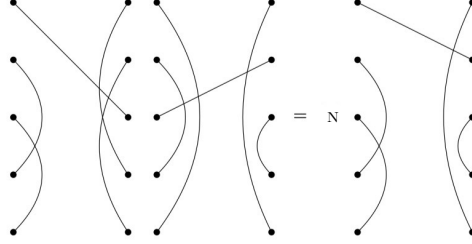


Figure 7: Example multiplication in the Brauer algebra. We trace the lines of two side-by-side matchings to obtain a new matching. The extra factor N incurred by a loop may in general be any complex number, but N is the relevant choice in this context (Figure 17 in [1]).

A special class $M_{n,n} \subset M_{2n}$ consists of matchings such that every opposite-side pairing is either between the top n vertices on the left and the top n vertices on the right, or between the bottom n vertices on the left and the bottom n vertices on the right. Moreover, every same-side pairing must be between one of the top n vertices and one of the bottom n vertices on the same side. The matchings in $M_{n,n}$ generate the **walled Brauer algebra** $\mathcal{B}_{n,n}$.

Of particular interest are the following elements in $M_{n,n} \subset \mathcal{B}_{n,n}$. For $i, j \in [n]$ or $i, j \in (n : 2n]$, let $\langle i, j \rangle$ be the matching that swaps i on the left with j on the right and j on the left with i on the right. For $i \in [n]$ and $j \in (n : 2n]$, let $\langle i, j \rangle$ be the matching that swaps i and j on the left and respectively on the right. See Figure 8.

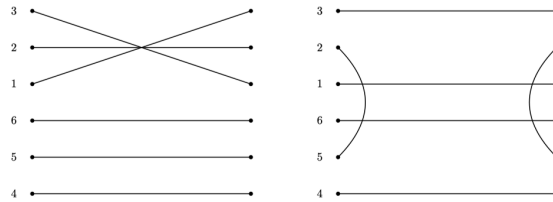


Figure 8: Left: $\langle 1, 3 \rangle$. Right: $\langle 2, 5 \rangle$ (Figure 20 in [1]).

As the colors indicate, the special matchings $\langle i, j \rangle$ and $\langle i, j \rangle$ exactly correspond to the green and blue lines we introduced to the strand diagram. We associate $(-1/N)\langle i, j \rangle$ to green swaps and $(1/N)\langle i, j \rangle$ to blue turnarounds and multiply these elements in the Brauer algebra as we read a strand diagram from left to right. The resulting element in the Brauer algebra is proportional to the matching obtained from tracing strands as in Figure 6.

Thus, given a collection of points $P \subset \mathcal{D}^T$, we get an element in $\mathcal{B}_{n,n}$ denoted $F(P)$ that is proportional to the matching $\pi(P)$ determined by P . Moreover, the coefficient in (3.3) correspond to the coefficient of $F(P)$ thanks to the factors of $\pm(1/N)$ we associate to the swaps and turnarounds and the factors of N resulting from loops in Brauer algebra multiplication. As a result, we get ((4.1) of [1])

$$\sum_{\pi \in M_{n,n}} w^T(\pi) \pi = e^{\binom{2n}{2}T - nT} \mathbb{E}[F(\Sigma^T)].$$

To prove (3.6), it suffices to understand $e^{\binom{2n}{2}T - nT} \mathbb{E}[F(\Sigma^T)]$ in the large T limit. We do not have space to cover the technical details in section 4 of [1]. Roughly speaking, in addition to calculating the correct coefficients, we need to understand why in the large T limit, the matchings in $M_{n,n}$ that survive are precisely those that only contain same-side pairs. The key Lemma 4.5 in [1] shows that before and after a turnaround $\langle i_0, j_0 \rangle$, one may ignore in expectation the other swaps and turnarounds that involves i_0 or j_0 , due to cancellation of signed elements. We are then able to recover $\mathbb{E}[F(\Sigma^T)]$ from certain exploration processes on the strand diagram that cannot see some of the swaps and turnarounds. The matchings discovered by these exploration processes only involve same-side pairs. Finally, the coefficients $\text{Wg}_N(\sigma\tau^{-1})$ are found by viewing the Weingarten function as an element in the group algebra $\mathbb{C}[S_n]$ and recovering its formula in terms of the so-called Jucys-Murphy elements in the large T limit.

Final Remarks

We did not cover section 5 of the paper, which uses the strand diagram formulation of Haar integrals and strand exploration techniques developed in section 4 to investigate Master loop equations. The authors are able to provide an alternative proof of the recursion formula for Wilson loop expectations previously studied in the literature. Section 6 generalizes the strand diagram and surface sum interpretations we have discussed for the unitary group to other classical Lie groups. The reader is encouraged to consult the original paper [1] for a list of further problems to investigate.

References

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